Quantum relations and zero-error communication

Dominic Verdon

Workshop on quantum graphs, Saarland University, February 2025

This talk is based on:

- Verdon, D.; A covariant Stinespring theorem. J. Math. Phys. 63 (9): 091705 [Ver22]
- Verdon, D.; Covariant Quantum Combinatorics with Applications to Zero-Error Communication. Commun. Math. Phys. 405, 51 (2024). [Ver24]
- Allen, R. and Verdon, D.; CP^{∞} and beyond: 2-categorical dilation theory. Th. Appl. Cat. Vol. 41, No. 50, pp 1783-1811 (2024) [AV24]

and some new work in production now.

A primer on 2-categories

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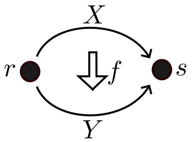

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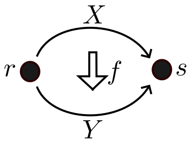

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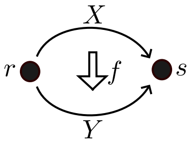

- 2-categories have a convenient planar diagrammatic calculus that handles composition of these different types of morphism.
- This generalises the 'tensor network/tensor diagram' calculus for tensor categories. A tensor category is precisely a 2-category with a single object.

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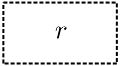
- We represent objects $r, s, \ldots$ as planar regions:



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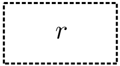

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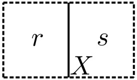

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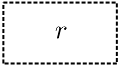

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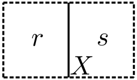

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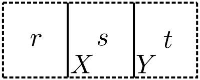

$$X \otimes Y : r \to t$$

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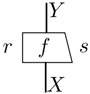

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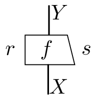

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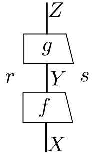

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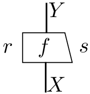

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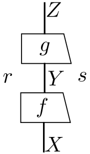

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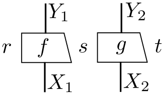

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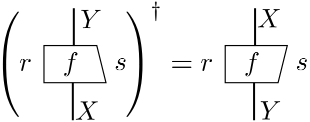

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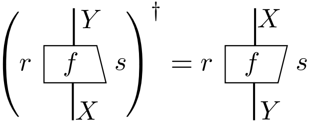

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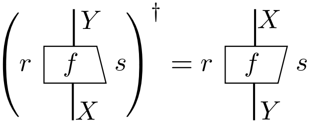

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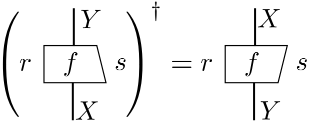

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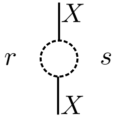

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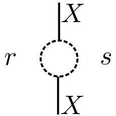

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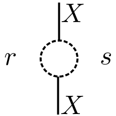

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- A C^*-2-category is a *W^*-2-category* if additionally every 2-morphism space $\mathrm{Hom}(X, Y)$ has a predual. In particular, the endomorphism algebras $\mathrm{End}(X)$ are W^*-algebras.

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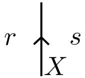

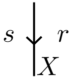

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- These satisfy the *snake* or *zigzag* equations:

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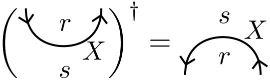

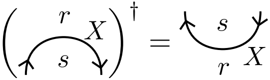

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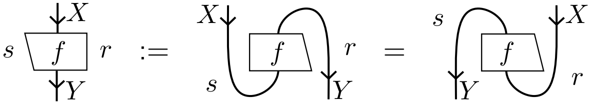

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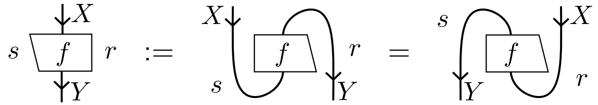
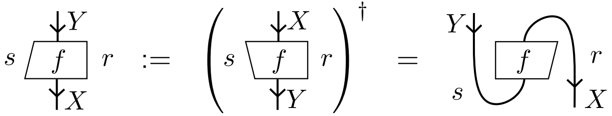
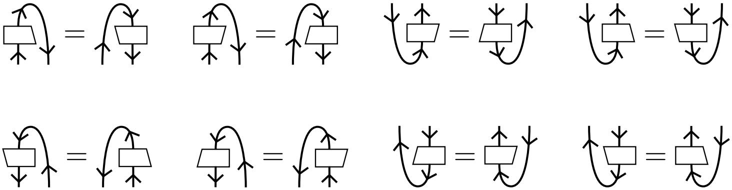




Sliding equations

We can slide the 2-morphisms around cups and caps:



Categorical quantum mechanics

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- Motivating example: the category $\mathrm{Rep}_{fd}(G)$ of finite-dimensional unitary representations of a compact quantum group G.

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- Example: rigid W^*-tensor categories.

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- If $\mathcal{T} \simeq \mathrm{Rep}_{fd}(G)$ for CQG G then $\mathrm{Mod}(\mathcal{T})$ is the finite-dimensional equivariant Morita 2-category:

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- See [DR18][CHPJP22][Ver22].

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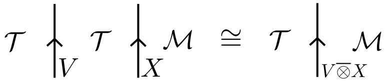

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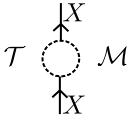

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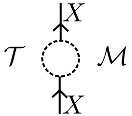

- If $\mathcal{T} \cong \mathrm{Rep}_{fd}(G)$, then $\mathrm{End}(X)$ is the fixed point subalgebra of the associated f.d. G-W^*-algebra.

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Definition (C.f [CH16])

Let $X : \mathcal{T} \to \mathcal{M}_1$ and $Y : \mathcal{T} \to \mathcal{M}_2$ be $\mathcal{T}$-systems. A *dilation* $X \to Y$ is a pair of:

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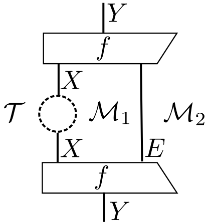
A dilation $X \to Y$ induces the following map $\mathrm{End}(X) \to \mathrm{End}(Y)$ on observables:



$\mathcal{T}$-dynamics: II

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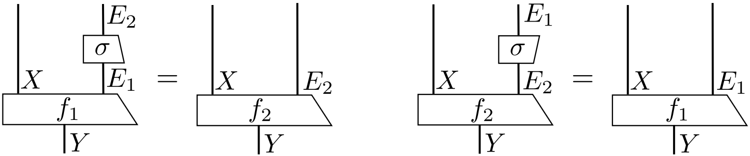

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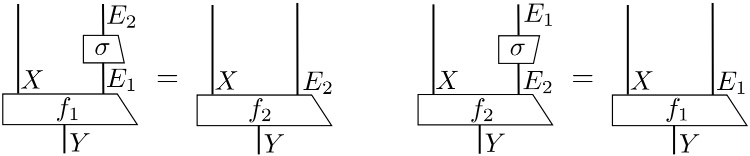

- We call a class of dilations $X \to Y$ under this equivalence relation a *CP map*.
- Every CP map has a *minimal dilation* which is related to all other dilations by an isometry.

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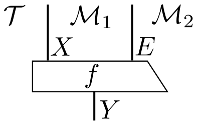
- Let $\Phi : X \to Y$ be a CP map with minimal dilation (E, f):



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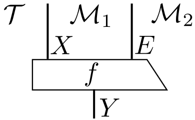

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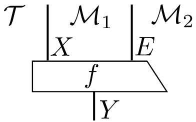

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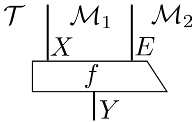

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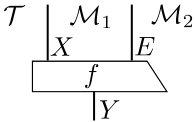

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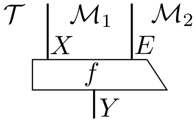

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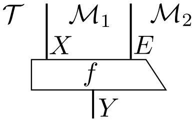

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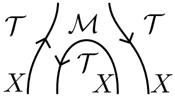
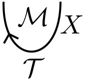
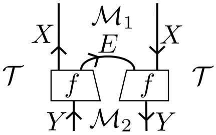
Q-systems

- A $\mathcal{T}$-system $X : \mathcal{T} \to \mathcal{M}$ induces an algebra $X \otimes X^*$ in $\mathcal{T}$, called a *Q-system*, whose multiplication and unit are defined using rigidity:





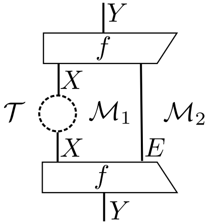
- Let $X : \mathcal{T} \to \mathcal{M}_1$, $Y : \mathcal{T} \to \mathcal{M}_2$ be $\mathcal{T}$-systems. A CP map $X \to Y$ can equivalently be defined as a map between Q-systems (c.f. [Sel07]):



- Q-system dynamics is in the Schrodinger picture: channels are trace preserving rather than unital.

Quantum relations

Quantum relations



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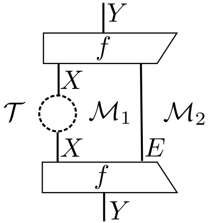

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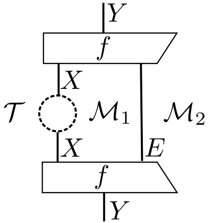

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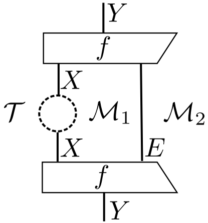

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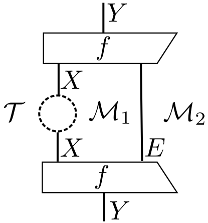

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- Forthcoming work: objects $\mathcal{M}$ can be interpreted as *equivalence classes of reference frames.* Simple objects of $\mathcal{M}$ are *classical types* associated with the frame.

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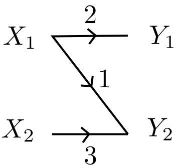

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Functors as abstract quantum relations

- The objects of $\mathrm{Mod}_{fd}(\mathcal{T})$ are *semisimple* module categories.
- A functor between semisimple categories is a lot like a relation:
  - Completely determined by the simple objects.
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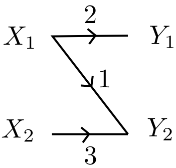

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- When $\mathcal{T} \simeq \mathrm{Rep}_{fd}(G)$ the 1-morphisms are equivariant bimodules (c.f. [Wea12]).

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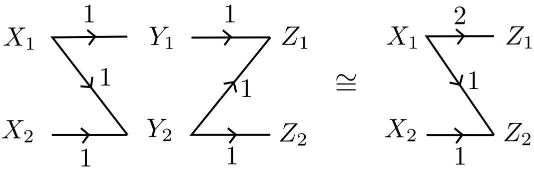

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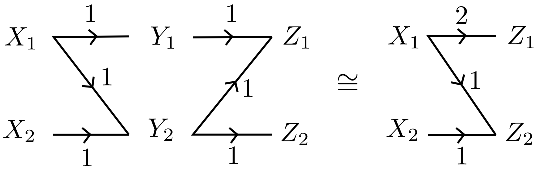
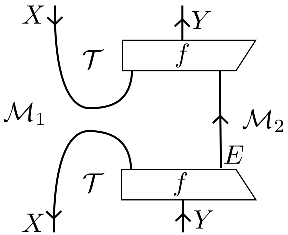
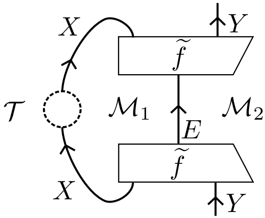


- To solve this problem we must take account of the 2-morphism structure.

Covariant Choi's theorem

- Using rigidity, CP maps $X \to Y$ can be identified with positive operators in the W^*-algebra $\mathrm{End}(X^* \otimes Y)$:





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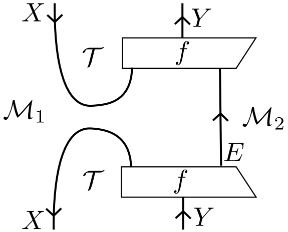

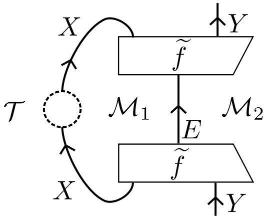

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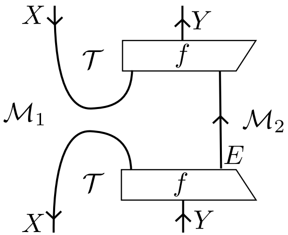

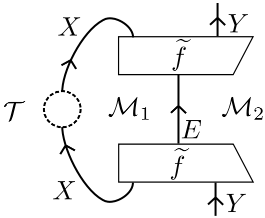

This is a bijective correspondence (in fact, an isomorphism of convex cones).

- Notation: If $\Phi : X \to Y$ is a CP map, then $\widetilde{\Phi} \in \mathrm{End}(X^* \otimes Y)$ is the associated positive operator.

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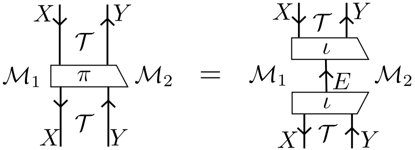

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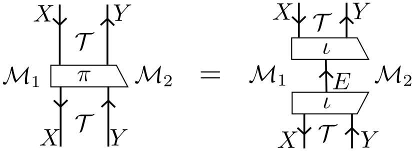

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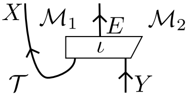

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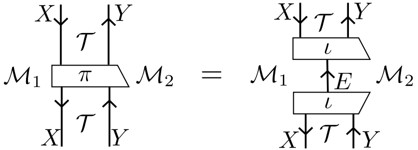

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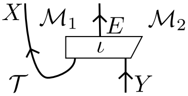

- C.f. [Kor20].

Underlying relations

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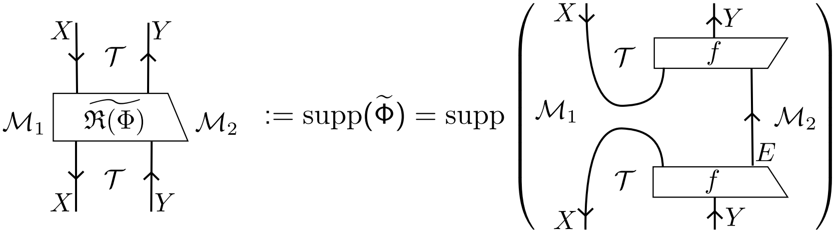

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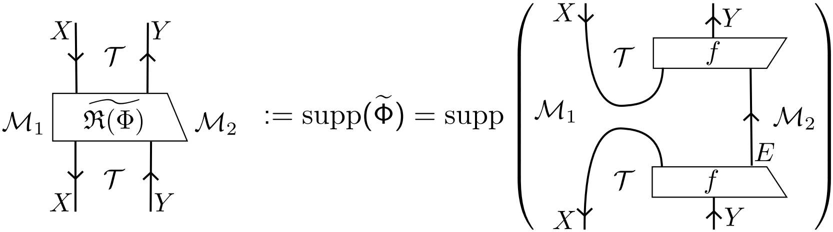

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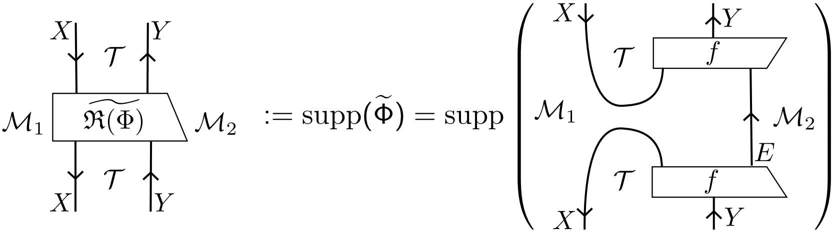

- The projection $\widetilde{\mathfrak{R}(\Phi)}$ splits to an isometry $\iota : E \to X^* \otimes Y$.
- The relation obtained does not depend on the choice of dilation.

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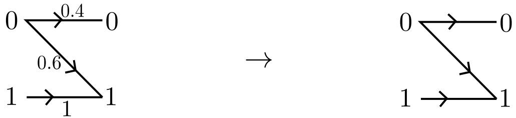

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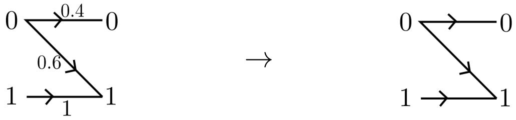

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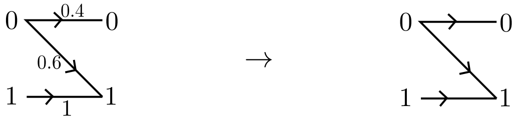

- The underlying quantum relation should tell us about zero-error properties, such as reversibility.
- We will first show functoriality of $\mathfrak{R}$.

The category of CP maps

- The category $\mathcal{T}\text{-CP}$ has:

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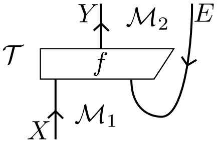
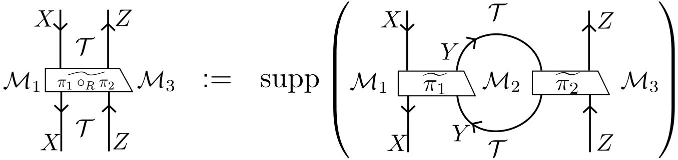
- There are subcategories $\mathcal{T}\text{-Alg} \subset \mathcal{T}\text{-Chan} \subset \mathcal{T}\text{-CP}$ whose morphisms are unital $*$-homomorphisms and channels respectively.
- $\mathcal{T}$-CP has a dagger structure which is *not* inherited by these two subcategories. This takes a CP map $X \to Y$ with dilation (E, f) to a CP map $Y \to X$ with the following dilation:



This is known as the *adjoint* of a CP map.

The category of quantum relations

- Let $\pi_1 : X \to Y$, $\pi_2 : Y \to Z$ be quantum relations. We define the composition $\pi_2 \circ_R \pi_1 : X \to Z$ as follows:



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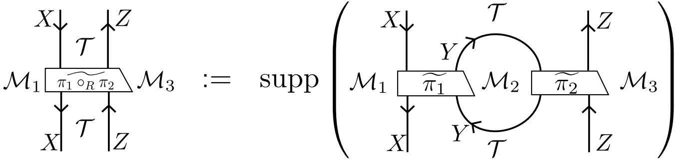

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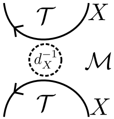

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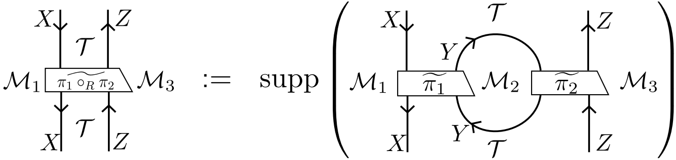

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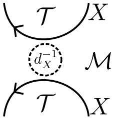
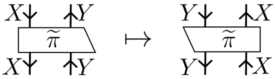


- The resulting category $\mathcal{T}$-Rel has a dagger — the *converse*:



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Proposition ([Ver24, Prop. 3.5])

There is a full and faithful identity-on-objects dagger functor $\mathfrak{R} : \mathcal{T}\text{-CP} \to \mathcal{T}\text{-Rel}$, *which sends a CP map to its underlying relation.*

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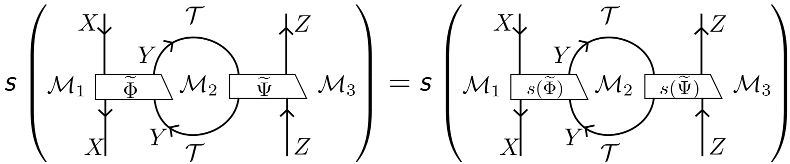

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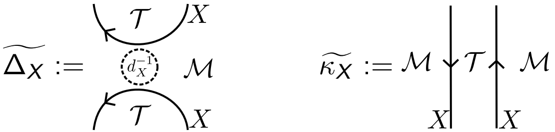

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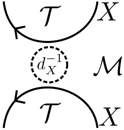
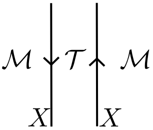
$\widetilde{\Delta_X} :=$ 

$\widetilde{\kappa_X} :=$ 

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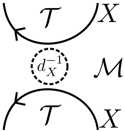
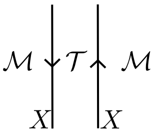
$\widetilde{\Delta_X} :=$ 

$\widetilde{\kappa_X} :=$ 

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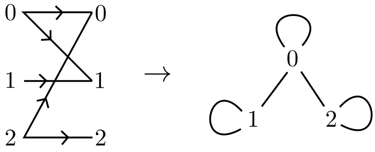
- The *complement* $\widetilde{\Gamma} \mapsto \widetilde{\kappa_X} - \widetilde{\Gamma}$ gives a bijective correspondence between simple and confusability graphs.

The confusability graph of a channel

In classical zero-error communication, we often need even less information than the underlying relation [Sha56]:



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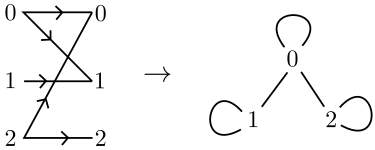

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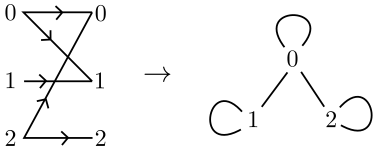

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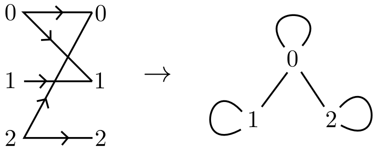

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- Alternatively, one can work with the simple complement, called the *distinguishability graph* [Sta16].

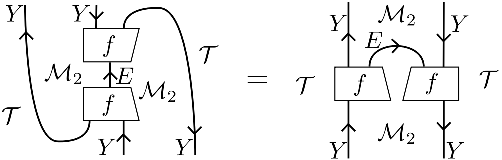
Duan's theorem

Theorem ([Ver24, Prop. 3.12], c.f.[Dua09, Lem. 2])

Let $\Gamma : Y \to Y$ be a confusability graph. Then there exists a channel $\Phi : X \to Y$ such that $\mathfrak{R}(\Phi \circ \Phi^\dagger) = \Gamma$.

Proof.

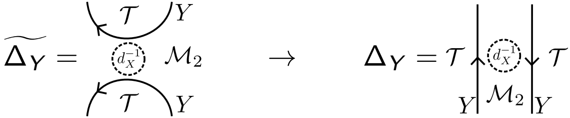
- $\widetilde{\Delta_Y} < \widetilde{\Gamma} \Rightarrow \widetilde{\Gamma} = \widetilde{\Delta_Y} + (\widetilde{\Gamma} - \widetilde{\Delta_Y})$. The summands are orthogonal projections; moreover they are symmetric.
- We use the Q-system version of Choi's theorem:



- We get an equality of CP maps $\Gamma = \Delta_Y + (\Gamma - \Delta_Y)$, where the summands are self-adjoint CP maps $Y \to Y$.

Duan's theorem

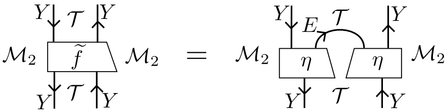
- We were considering the equality of CP maps $\Gamma = \Delta_Y + (\Gamma - \Delta_Y)$.
- We have the following concrete expression for Δ_Y:



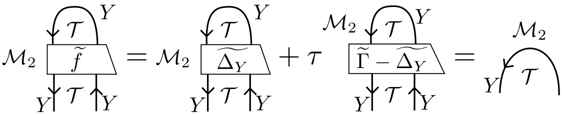
- We observe that Δ_Y is a positive invertible operator in $\mathrm{End}(Y \otimes Y^*)$.
- Therefore for small enough $\tau > 0$, the CP map $f := \Delta_Y + \tau(\Gamma - \Delta_Y)$ is positive.
- We now go back to the Choi space and consider the positive operator $\widetilde{f} = \widetilde{\Delta_Y} + \tau(\widetilde{\Gamma} - \widetilde{\Delta_Y}) \in \mathrm{End}(Y^* \otimes Y)$.

Duan's theorem

- We were considering the positive operator $\widetilde{f} = \widetilde{\Delta_Y} + \tau(\widetilde{\Gamma} - \widetilde{\Delta_Y}) \in \mathrm{End}(Y^* \otimes Y)$.
- Fact 1: Positivity of f implies that $\widetilde{f}$ is CP, in the sense that it has a dilation $\eta : X^* \to X^* \otimes E$:

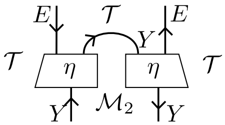


- Fact 2: $\widetilde{f}$ is furthermore a channel:

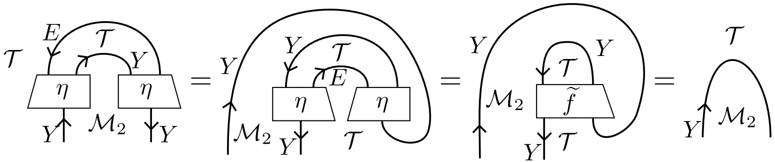


- Fact 3: $s(\widetilde{f}) = s(\widetilde{\Delta_Y} + \tau(\widetilde{\Gamma} - \widetilde{\Delta_Y})) = s(\widetilde{\Gamma}) = \widetilde{\Gamma}$.

- We were discussing the positive operator $\widetilde{f} \in \mathrm{End}(Y^* \otimes Y)$. It has a dilation $\eta : Y^* \to Y^* \otimes E$.
- We define the following CP map $\Phi : E^* \to Y$:



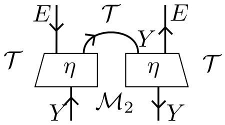
- Φ is a channel:



- We will finish by showing that its quantum confusability graph is Γ.

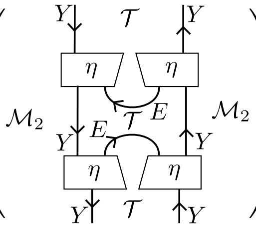
Duan's theorem

- We just defined a channel $\Phi : E^* \to Y$:



- We have

$$\mathfrak{R}(\widetilde{\Phi \circ \Phi^\dagger}) = s\left(\quad \right)$$



$$= s(\widetilde{f}^\dagger \circ \widetilde{f}) = s(\widetilde{f}) = \widetilde{\Gamma}.$$

$\square$

More on quantum relations and zero error communication

Theorem (Reversibility of channels)

[[Ver24, Thm. 4.4]] Let $\Phi : Y \to X$ be a channel. There exists a channel $\Psi : X \to Y$ such that $\Phi \circ \Psi = \mathrm{id}_X$ iff the confusability graph of Φ is discrete.

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To state the following theorem we needed $\mathcal{T}$ to be a braided tensor category to obtain a tensor product of $\mathcal{T}$-systems.

Theorem (Source channel coding)

[[Ver24, Thm. 5.3], c.f. [Sta16])] A covariant channel is a valid encoding channel for a zero-error source-channel coding scheme precisely when it is a homomorphism from the confusability graph of the source to the confusability graph of the communication channel.

Infinite dimension: outlook

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- More generally, category of unitary representations of a locally compact quantum group?

Free cocompletion

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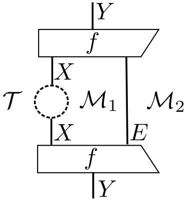

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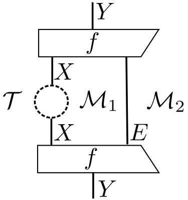

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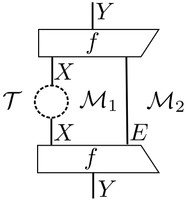

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- So dynamics again splits into vertical and horizontal levels.

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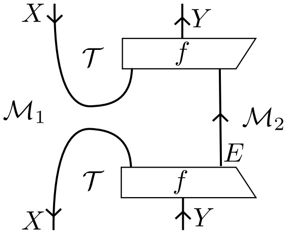

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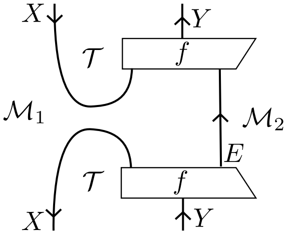

- This depends on rigidity, and therefore so does our subsequent analysis.

Thanks for listening!

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