Entanglement-symmetries of covariant channels

arXiv:2012.05761

Dominic Verdon
University of Bristol

University of York
February 27, 2024

E-A channel coding

Channels

- We will call a completely positive trace-preserving map $f : A \to B$ between finite-dimensional (f.d.) C^*-algebras a *channel*.

Channels

- We will call a completely positive trace-preserving map $f : A \to B$ between finite-dimensional (f.d.) C^*-algebras a *channel*.
- This is the standard notion of a dynamical map in quantum information theory.

Channels

- We will call a completely positive trace-preserving map $f : A \to B$ between finite-dimensional (f.d.) C^*-algebras a *channel*.
- This is the standard notion of a dynamical map in quantum information theory.
- Channels $B(H) \to B(K)$ between algebras of operators on f.d. Hilbert spaces are known as *quantum-to-quantum channels*.

Channels

- We will call a completely positive trace-preserving map $f : A \to B$ between finite-dimensional (f.d.) C^*-algebras a *channel*.
- This is the standard notion of a dynamical map in quantum information theory.
- Channels $B(H) \to B(K)$ between algebras of operators on f.d. Hilbert spaces are known as *quantum-to-quantum channels*.
- Channels $B(H) \to \mathbb{C}^{\oplus n}$ are called *quantum-to-classical channels*, or *POVMs* (positive operator valued measurements). They are determined by a family of positive operators $\{M_i \in B(H)\}_{i \in \{1,\dots,n\}}$.

Channels

- We will call a completely positive trace-preserving map $f : A \to B$ between finite-dimensional (f.d.) C^*-algebras a *channel*.
- This is the standard notion of a dynamical map in quantum information theory.
- Channels $B(H) \to B(K)$ between algebras of operators on f.d. Hilbert spaces are known as *quantum-to-quantum channels*.
- Channels $B(H) \to \mathbb{C}^{\oplus n}$ are called *quantum-to-classical channels*, or *POVMs* (positive operator valued measurements). They are determined by a family of positive operators $\{M_i \in B(H)\}_{i \in \{1,\dots,n\}}$.
- Channels $\mathbb{C}^{\oplus n} \to B(H)$ are called *classical-to-quantum channels*. They are determined by a family of states $\{\rho_i \in B(H)\}_{i \in \{1,\dots,n\}}$.

Entanglement-assisted channel coding

- Alice and Bob share a communication channel $N : A \to B$ and a maximally entangled state $\Psi : \mathbb{C} \to B(V) \otimes B(V)$.

Entanglement-assisted channel coding

- Alice and Bob share a communication channel $N : A \to B$ and a maximally entangled state $\Psi : \mathbb{C} \to B(V) \otimes B(V)$.
- They want to communicate through a channel $T : X \to Y$.

Entanglement-assisted channel coding

- Alice and Bob share a communication channel $N : A \to B$ and a maximally entangled state $\Psi : \mathbb{C} \to B(V) \otimes B(V)$.
- They want to communicate through a channel $T : X \to Y$.
- To achieve this, Alice performs an encoding channel $E : X \otimes B(V) \to A$ using her half of the entangled state, and transmits the resulting state of A using the channel $N : A \to B$.

Entanglement-assisted channel coding

- Alice and Bob share a communication channel $N : A \to B$ and a maximally entangled state $\Psi : \mathbb{C} \to B(V) \otimes B(V)$.
- They want to communicate through a channel $T : X \to Y$.
- To achieve this, Alice performs an encoding channel $E : X \otimes B(V) \to A$ using her half of the entangled state, and transmits the resulting state of A using the channel $N : A \to B$.
- Bob then performs a decoding channel $D : B \otimes B(V) \to Y$ using his half of the entangled state.

Entanglement-assisted channel coding

- Alice and Bob share a communication channel $N : A \to B$ and a maximally entangled state $\Psi : \mathbb{C} \to B(V) \otimes B(V)$.
- They want to communicate through a channel $T : X \to Y$.
- To achieve this, Alice performs an encoding channel $E : X \otimes B(V) \to A$ using her half of the entangled state, and transmits the resulting state of A using the channel $N : A \to B$.
- Bob then performs a decoding channel $D : B \otimes B(V) \to Y$ using his half of the entangled state.
- We say that (E, D, V) is an *entanglement-assisted channel coding scheme* for T from N if the resulting channel

$$D \circ (N \otimes \mathrm{id}_{B(V)}) \circ (E \otimes \mathrm{id}_{B(V)}) \circ (\mathrm{id}_X \otimes \Psi) : X \to Y$$

from Alice to Bob is equal to T.

Entanglement-equivalent channels

- The relation

$$(N_1 : A_1 \to B_1) \geq (N_2 : A_2 \to B_2)$$
$$\text{iff there exists an entanglement} - \text{assisted channel}$$
$$\text{coding scheme for } N_2 \text{ from } N_1$$

defines a partial order on channels.

Entanglement-equivalent channels

- The relation

$$(N_1 : A_1 \to B_1) \geq (N_2 : A_2 \to B_2)$$
$$\text{iff there exists an entanglement} - \text{assisted channel}$$
$$\text{coding scheme for } N_2 \text{ from } N_1$$

  defines a partial order on channels.
- We are interested in classes of channels that are equivalent under this partial order; that is, channels which can simulate each other using an entangled resource. We call such channels *entanglement-equivalent*.

Entanglement-equivalent channels

- The relation

$$(N_1 : A_1 \to B_1) \geq (N_2 : A_2 \to B_2)$$
$$\text{iff there exists an entanglement} - \text{assisted channel coding scheme for } N_2 \text{ from } N_1$$

  defines a partial order on channels.
- We are interested in classes of channels that are equivalent under this partial order; that is, channels which can simulate each other using an entangled resource. We call such channels *entanglement-equivalent*.
- For a similar problem without entanglement, see M.B. Hastings, 'Infinitely many kinds of quantum channels' (2008).

Our results

- Given a channel that possesses some symmetry properties (called ‘covariance’), we will present a construction of other channels which are entanglement-equivalent to it.

Our results

- Given a channel that possesses some symmetry properties (called ‘covariance’), we will present a construction of other channels which are entanglement-equivalent to it.
- This construction does not solve the problem of determining whether a pair of channels are entanglement-equivalent.

Our results

- Given a channel that possesses some symmetry properties (called ‘covariance’), we will present a construction of other channels which are entanglement-equivalent to it.
- This construction does not solve the problem of determining whether a pair of channels are entanglement-equivalent.
- However, it represents a first step in this direction.

Our results

- Given a channel that possesses some symmetry properties (called 'covariance'), we will present a construction of other channels which are entanglement-equivalent to it.
- This construction does not solve the problem of determining whether a pair of channels are entanglement-equivalent.
- However, it represents a first step in this direction.
- As a first application, we will show how the construction can be used to compute the entanglement-assisted capacities of certain quantum channels.

Covariance of channels

Covariance for channels

- *Covariance* is the standard way to define what we mean when we say that a channel possesses some symmetry.

Covariance for channels

- *Covariance* is the standard way to define what we mean when we say that a channel possesses some symmetry.
- Let us fix a compact symmetry group G.

Covariance for channels

- *Covariance* is the standard way to define what we mean when we say that a channel possesses some symmetry.
- Let us fix a compact symmetry group G.
- Suppose we have two f.d. C^*-algebras A, B, respectively carrying *actions* of G; that is, continuous group homomorphisms

$$\pi_A : G \to \mathrm{Aut}(A) \qquad \pi_B : G \to \mathrm{Aut}(B).$$

Covariance for channels

- *Covariance* is the standard way to define what we mean when we say that a channel possesses some symmetry.
- Let us fix a compact symmetry group G.
- Suppose we have two f.d. C^*-algebras A, B, respectively carrying *actions* of G; that is, continuous group homomorphisms

$$\pi_A : G \to \mathrm{Aut}(A) \qquad \pi_B : G \to \mathrm{Aut}(B).$$

- We say that a channel $f : A \to B$ is *covariant* for these actions when:

$$\pi_B(g) \circ f = f \circ \pi_A(g) \qquad \text{for all } g \in G$$

From groups to Hopf algebras

- Using the duality between topological spaces and algebras of continuous functions on those spaces, we can exchange the compact group G for a certain *Hopf $*$-algebra* $\mathbb{C}[G]$.

From groups to Hopf algebras

- Using the duality between topological spaces and algebras of continuous functions on those spaces, we can exchange the compact group G for a certain *Hopf $*$-algebra* $\mathbb{C}[G]$.
- This is a commutative unital $*$-algebra equipped with a unital $*$-homomorphism

$$\Delta : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$$

(the *comultiplication*) and two linear maps

$$\epsilon : \mathbb{C}[G] \to \mathbb{C} \qquad\qquad S : \mathbb{C}[G] \to \mathbb{C}[G]$$

which we call the *counit* and *antipode* respectively. This data satisfies certain equations.

From groups to Hopf algebras

- Using the duality between topological spaces and algebras of continuous functions on those spaces, we can exchange the compact group G for a certain *Hopf $*$-algebra* $\mathbb{C}[G]$.
- This is a commutative unital $*$-algebra equipped with a unital $*$-homomorphism

$$\Delta : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$$

(the *comultiplication*) and two linear maps

$$\epsilon : \mathbb{C}[G] \to \mathbb{C} \qquad\qquad S : \mathbb{C}[G] \to \mathbb{C}[G]$$

which we call the *counit* and *antipode* respectively. This data satisfies certain equations.

- It is a subalgebra of the algebra $C(G)$ of all continuous complex-valued functions on the compact group G.

The Hopf-algebraic formulation of covariance

- Actions of the group G on a f.d. C^*-algebra A correspond to *coactions* of the Hopf $*$-algebra $\mathbb{C}[G]$ on A.

The Hopf-algebraic formulation of covariance

- Actions of the group G on a f.d. C^*-algebra A correspond to *coactions* of the Hopf $*$-algebra $\mathbb{C}[G]$ on A.
- These are unital $*$-homomorphisms $\alpha : A \to A \otimes \mathbb{C}[G]$ satisfying certain equations.

The Hopf-algebraic formulation of covariance

- Actions of the group G on a f.d. C^*-algebra A correspond to *coactions* of the Hopf $*$-algebra $\mathbb{C}[G]$ on A.
- These are unital $*$-homomorphisms $\alpha : A \to A \otimes \mathbb{C}[G]$ satisfying certain equations.
- With respect to coactions α_A, α_B on f.d. C^*-algebras A, B, covariance of a channel $f : A \to B$ comes down to the following equation:

$$(f \otimes \mathrm{id}_{\mathbb{C}[G]}) \circ \alpha_A = \alpha_B \circ f$$

Compact quantum group algebras

- We generalise from commutative to possibly noncommutative Hopf $*$-algebras.

Compact quantum group algebras

- We generalise from commutative to possibly noncommutative Hopf $*$-algebras.
- We will call a Hopf $*$-algebra (obeying a minor technical condition) a *compact quantum group algebra*. It can be thought of as an algebra of continuous functions on a 'compact quantum group'.

Compact quantum group algebras

- We generalise from commutative to possibly noncommutative Hopf $*$-algebras.
- We will call a Hopf $*$-algebra (obeying a minor technical condition) a *compact quantum group algebra*. It can be thought of as an algebra of continuous functions on a 'compact quantum group'.
- Any commutative compact quantum group algebras is isomorphic to $\mathbb{C}[G]$ for some compact group G.

Compact quantum group algebras

- We generalise from commutative to possibly noncommutative Hopf $*$-algebras.
- We will call a Hopf $*$-algebra (obeying a minor technical condition) a *compact quantum group algebra*. It can be thought of as an algebra of continuous functions on a 'compact quantum group'.
- Any commutative compact quantum group algebras is isomorphic to $\mathbb{C}[G]$ for some compact group G.
- Coactions on f.d. C^*-algebras and covariance of channels can be defined just as in the commutative case.

Compact quantum group algebras

- We generalise from commutative to possibly noncommutative Hopf $*$-algebras.
- We will call a Hopf $*$-algebra (obeying a minor technical condition) a *compact quantum group algebra*. It can be thought of as an algebra of continuous functions on a 'compact quantum group'.
- Any commutative compact quantum group algebras is isomorphic to $\mathbb{C}[G]$ for some compact group G.
- Coactions on f.d. C^*-algebras and covariance of channels can be defined just as in the commutative case.
- This generalisation is necessary! We shall see why shortly.

The category of covariant channels

- Recall that a *category* $\mathcal{C}$ is defined by the following data:

The category of covariant channels

- Recall that a *category* $\mathcal{C}$ is defined by the following data:
  - A set of *objects* $X, Y, \ldots$.

The category of covariant channels

- Recall that a *category* $\mathcal{C}$ is defined by the following data:
  - A set of *objects* $X, Y, \ldots$.
  - For every ordered pair of objects (X, Y), a set of *morphisms* $\mathrm{Hom}(X, Y)$. (These can be thought of as 'arrows' between the objects, where X is the source and Y is the target.)

The category of covariant channels

- Recall that a *category* $\mathcal{C}$ is defined by the following data:
  - A set of *objects* $X, Y, \ldots$.
  - For every ordered pair of objects (X, Y), a set of *morphisms* $\mathrm{Hom}(X, Y)$. (These can be thought of as 'arrows' between the objects, where X is the source and Y is the target.)
  - Morphisms with compatible source and target can be composed; i.e. for $f : X \to Y$ and $g : Y \to Z$ we can define $g \circ f : X \to Z$.

The category of covariant channels

- Recall that a *category* $\mathcal{C}$ is defined by the following data:
  - A set of *objects* $X, Y, \ldots$.
  - For every ordered pair of objects (X, Y), a set of *morphisms* $\mathrm{Hom}(X, Y)$. (These can be thought of as 'arrows' between the objects, where X is the source and Y is the target.)
  - Morphisms with compatible source and target can be composed; i.e. for $f : X \to Y$ and $g : Y \to Z$ we can define $g \circ f : X \to Z$.
- This data must obey the following conditions:

The category of covariant channels

- Recall that a *category* $\mathcal{C}$ is defined by the following data:
  - A set of *objects* $X, Y, \ldots$.
  - For every ordered pair of objects (X, Y), a set of *morphisms* $\mathrm{Hom}(X, Y)$. (These can be thought of as 'arrows' between the objects, where X is the source and Y is the target.)
  - Morphisms with compatible source and target can be composed; i.e. for $f : X \to Y$ and $g : Y \to Z$ we can define $g \circ f : X \to Z$.
- This data must obey the following conditions:
  - Every object X possesses an *identity* morphism $\mathrm{id}_X : X \to X$. Composing with an identity morphism does nothing.

The category of covariant channels

- Recall that a *category* $\mathcal{C}$ is defined by the following data:
  - A set of *objects* $X, Y, \ldots$.
  - For every ordered pair of objects (X, Y), a set of *morphisms* $\mathrm{Hom}(X, Y)$. (These can be thought of as 'arrows' between the objects, where X is the source and Y is the target.)
  - Morphisms with compatible source and target can be composed; i.e. for $f : X \to Y$ and $g : Y \to Z$ we can define $g \circ f : X \to Z$.
- This data must obey the following conditions:
  - Every object X possesses an *identity* morphism $\mathrm{id}_X : X \to X$. Composing with an identity morphism does nothing.
  - Composition of morphisms is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.

The category of covariant channels

- Recall that a *category* $\mathcal{C}$ is defined by the following data:
  - A set of *objects* $X, Y, \ldots$.
  - For every ordered pair of objects (X, Y), a set of *morphisms* $\mathrm{Hom}(X, Y)$. (These can be thought of as 'arrows' between the objects, where X is the source and Y is the target.)
  - Morphisms with compatible source and target can be composed; i.e. for $f : X \to Y$ and $g : Y \to Z$ we can define $g \circ f : X \to Z$.
- This data must obey the following conditions:
  - Every object X possesses an *identity* morphism $\mathrm{id}_X : X \to X$. Composing with an identity morphism does nothing.
  - Composition of morphisms is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.
- For any compact quantum group algebra H there is a category $\mathrm{Chan}(H)$ where:

The category of covariant channels

- Recall that a *category* $\mathcal{C}$ is defined by the following data:
  - A set of *objects* $X, Y, \ldots$.
  - For every ordered pair of objects (X, Y), a set of *morphisms* $\mathrm{Hom}(X, Y)$. (These can be thought of as 'arrows' between the objects, where X is the source and Y is the target.)
  - Morphisms with compatible source and target can be composed; i.e. for $f : X \to Y$ and $g : Y \to Z$ we can define $g \circ f : X \to Z$.
- This data must obey the following conditions:
  - Every object X possesses an *identity* morphism $\mathrm{id}_X : X \to X$. Composing with an identity morphism does nothing.
  - Composition of morphisms is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.
- For any compact quantum group algebra H there is a category $\mathrm{Chan}(H)$ where:
  - *Objects* are f.d. C^*-algebras equipped with an H-coaction.

The category of covariant channels

- Recall that a *category* $\mathcal{C}$ is defined by the following data:
  - A set of *objects* $X, Y, \ldots$.
  - For every ordered pair of objects (X, Y), a set of *morphisms* $\mathrm{Hom}(X, Y)$. (These can be thought of as 'arrows' between the objects, where X is the source and Y is the target.)
  - Morphisms with compatible source and target can be composed; i.e. for $f : X \to Y$ and $g : Y \to Z$ we can define $g \circ f : X \to Z$.
- This data must obey the following conditions:
  - Every object X possesses an *identity* morphism $\mathrm{id}_X : X \to X$. Composing with an identity morphism does nothing.
  - Composition of morphisms is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.
- For any compact quantum group algebra H there is a category $\mathrm{Chan}(H)$ where:
  - *Objects* are f.d. C^*-algebras equipped with an H-coaction.
  - *Morphisms* are covariant channels.

Entanglement-symmetries

Hopf-Galois objects

- Let H be a compact quantum group algebra.

Hopf-Galois objects

- Let H be a compact quantum group algebra.
- Associated to H are certain *Hopf-Galois objects*, which can be thought of as 'noncommutative torsors' for the compact quantum group.

Hopf-Galois objects

- Let H be a compact quantum group algebra.
- Associated to H are certain *Hopf-Galois objects*, which can be thought of as ‘noncommutative torsors’ for the compact quantum group.
- These Hopf-Galois objects are unital $*$-algebras $X, Y, \ldots$ equipped with unital $*$-homomorphisms $\alpha : X \to X \otimes H$, satisfying certain equations.

From Hopf-Galois objects to functors

Recall that a *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories is a map from objects to objects and morphisms to morphisms that

From Hopf-Galois objects to functors

Recall that a *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories is a map from objects to objects and morphisms to morphisms that

- is compatible with composition: for morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we have $F(g) \circ F(f) = F(g \circ f) : F(X) \to F(Z)$.

From Hopf-Galois objects to functors

Recall that a *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories is a map from objects to objects and morphisms to morphisms that

- is compatible with composition: for morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we have $F(g) \circ F(f) = F(g \circ f) : F(X) \to F(Z)$.
- respects identities: $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

From Hopf-Galois objects to functors

Recall that a *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories is a map from objects to objects and morphisms to morphisms that

- is compatible with composition: for morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we have $F(g) \circ F(f) = F(g \circ f) : F(X) \to F(Z)$.
- respects identities: $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Lemma

Let H be a compact quantum group algebra and let X be a Hopf-Galois object for H. Then we obtain

From Hopf-Galois objects to functors

Recall that a *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories is a map from objects to objects and morphisms to morphisms that

- is compatible with composition: for morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we have $F(g) \circ F(f) = F(g \circ f) : F(X) \to F(Z)$.
- respects identities: $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Lemma

Let H be a compact quantum group algebra and let X be a Hopf-Galois object for H. Then we obtain

- *A new compact quantum group algebra H^X. (Note that even if H was commutative H^X need not be commutative; this was why we needed to generalise to compact quantum groups.)*

From Hopf-Galois objects to functors

Recall that a *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories is a map from objects to objects and morphisms to morphisms that

- is compatible with composition: for morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we have $F(g) \circ F(f) = F(g \circ f) : F(X) \to F(Z)$.
- respects identities: $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Lemma

Let H be a compact quantum group algebra and let X be a Hopf-Galois object for H. Then we obtain

- *A new compact quantum group algebra H^X. (Note that even if H was commutative H^X need not be commutative; this was why we needed to generalise to compact quantum groups.)*
- *A functor $F_X : \mathrm{Chan}(H) \to \mathrm{Chan}(H^X)$.*

Entanglement-symmetries: I

Let $\Psi : \mathbb{C} \to B(V) \otimes B(V)$ be the channel initialising a maximally entangled state.

Entanglement-symmetries: I

Let $\Psi : \mathbb{C} \to B(V) \otimes B(V)$ be the channel initialising a maximally entangled state.

Theorem

Let H be a compact quantum group algebra, let X be a Hopf-Galois object for H, and let $\pi : X \to B(V)$ be a $*$-representation of X on a f.d. Hilbert space V.

Entanglement-symmetries: I

Let $\Psi : \mathbb{C} \to B(V) \otimes B(V)$ be the channel initialising a maximally entangled state.

Theorem

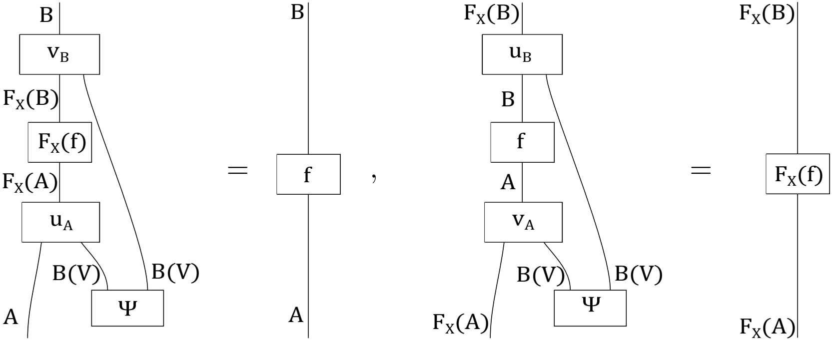
Let H be a compact quantum group algebra, let X be a Hopf-Galois object for H, and let $\pi : X \to B(V)$ be a $*$-representation of X on a f.d. Hilbert space V. Then for every object A of $\mathrm{Chan}(H)$ we obtain a pair of channels

$$u_A : A \otimes B(V) \to F_X(A) \qquad v_A : F_X(A) \otimes B(V) \to A$$

Entanglement-symmetries: II

Theorem (Continued)

For every H-covariant channel $f : A \to B$, the following equations are obeyed:



Entanglement-symmetries of covariant channels

- We call these transformations arising from f.d. $*$-representations of Hopf-Galois objects *entanglement-symmetries of covariant channels*.

Entanglement-symmetries of covariant channels

- We call these transformations arising from f.d. $*$-representations of Hopf-Galois objects *entanglement-symmetries of covariant channels*.
- They are symmetries, not of a single covariant channel alone, but of the whole category $\mathrm{Chan}(H)$.

Entanglement-symmetries of covariant channels

- We call these transformations arising from f.d. $*$-representations of Hopf-Galois objects *entanglement-symmetries of covariant channels*.
- They are symmetries, not of a single covariant channel alone, but of the whole category $\mathrm{Chan}(H)$.
- We observe in particular that for
  - any H-covariant channel $f : A \to B$
  - and any Hopf-Galois object X for H with a f.d. $*$-representation

  the H^X-covariant channel $F_X(f) : F_X(A) \to F_X(B)$ is entanglement-equivalent to f.

Examples

Twisted group algebras

- Let G be any finite group.

Twisted group algebras

- Let G be any finite group.
- We will consider a class of f.d. C^*-algebras called *twisted group algebras* $A(L, \phi)$.

Twisted group algebras

- Let G be any finite group.
- We will consider a class of f.d. C^*-algebras called *twisted group algebras* $A(L, \phi)$.These are defined by:
  - A subgroup $L < G$.

Twisted group algebras

- Let G be any finite group.
- We will consider a class of f.d. C^*-algebras called *twisted group algebras* $A(L, \phi)$. These are defined by:
  - A subgroup $L < G$.
  - A 2-cocycle $\phi \in Z^2(L, U(1))$.

Twisted group algebras

- Let G be any finite group.
- We will consider a class of f.d. C^*-algebras called *twisted group algebras* $A(L, \phi)$.These are defined by:
  - A subgroup $L < G$.
  - A 2-cocycle $\phi \in Z^2(L, U(1))$.
- They have a basis $\{u_g \mid g \in L\}$, which is orthogonal w.r.t. the Hilbert-Schmidt inner product.

Twisted group algebras

- Let G be any finite group.
- We will consider a class of f.d. C^*-algebras called *twisted group algebras* $A(L, \phi)$.These are defined by:
  - A subgroup $L < G$.
  - A 2-cocycle $\phi \in Z^2(L, U(1))$.
- They have a basis $\{u_g \mid g \in L\}$, which is orthogonal w.r.t. the Hilbert-Schmidt inner product.
- The multiplication, unit and involution are defined as follows:

Twisted group algebras

- Let G be any finite group.
- We will consider a class of f.d. C^*-algebras called *twisted group algebras* $A(L, \phi)$.These are defined by:
  - A subgroup $L < G$.
  - A 2-cocycle $\phi \in Z^2(L, U(1))$.
- They have a basis $\{u_g \mid g \in L\}$, which is orthogonal w.r.t. the Hilbert-Schmidt inner product.
- The multiplication, unit and involution are defined as follows:
  - Multiplication: $u_g \cdot u_h := \phi(g, h)u_{gh}$.

Twisted group algebras

- Let G be any finite group.
- We will consider a class of f.d. C^*-algebras called *twisted group algebras* $A(L, \phi)$.These are defined by:
  - A subgroup $L < G$.
  - A 2-cocycle $\phi \in Z^2(L, U(1))$.
- They have a basis $\{u_g \mid g \in L\}$, which is orthogonal w.r.t. the Hilbert-Schmidt inner product.
- The multiplication, unit and involution are defined as follows:
  - Multiplication: $u_g \cdot u_h := \phi(g, h) u_{gh}$.
  - Unit: u_e.

Twisted group algebras

- Let G be any finite group.
- We will consider a class of f.d. C^*-algebras called *twisted group algebras* $A(L, \phi)$. These are defined by:
  - A subgroup $L < G$.
  - A 2-cocycle $\phi \in Z^2(L, U(1))$.
- They have a basis $\{u_g \mid g \in L\}$, which is orthogonal w.r.t. the Hilbert-Schmidt inner product.
- The multiplication, unit and involution are defined as follows:
  - Multiplication: $u_g \cdot u_h := \phi(g, h) u_{gh}$.
  - Unit: u_e.
  - Involution: $u_g^\dagger := u_{g^{-1}}$.

Twisted group algebras

- Let G be any finite group.
- We will consider a class of f.d. C^*-algebras called *twisted group algebras* $A(L, \phi)$.These are defined by:
  - A subgroup $L < G$.
  - A 2-cocycle $\phi \in Z^2(L, U(1))$.
- They have a basis $\{u_g \mid g \in L\}$, which is orthogonal w.r.t. the Hilbert-Schmidt inner product.
- The multiplication, unit and involution are defined as follows:
  - Multiplication: $u_g \cdot u_h := \phi(g, h)u_{gh}$.
  - Unit: u_e.
  - Involution: $u_g^\dagger := u_{g^{-1}}$.
- We call a channel $f : A(L_1, \phi_1) \to A(L_2, \phi_2)$ between these twisted group algebras *covariant* if $f(u_g) = \lambda_g u_g$, for $\{\lambda_g \in \mathbb{C}\}_{g \in L_1}$.

Hopf-Galois objects

- Twisted group algebras, and covariant channels between them, are part of $\mathrm{Chan}(H)$ for a compact quantum group algebra H.

Hopf-Galois objects

- Twisted group algebras, and covariant channels between them, are part of $\mathrm{Chan}(H)$ for a compact quantum group algebra H.
- Hopf-Galois objects for H correspond to 2-cohomology classes $[\psi] \in H^2(G, U(1))$.

Hopf-Galois objects

- Twisted group algebras, and covariant channels between them, are part of $\mathrm{Chan}(H)$ for a compact quantum group algebra H.
- Hopf-Galois objects for H correspond to 2-cohomology classes $[\psi] \in H^2(G, U(1))$.
- The new compact quantum group algebra corresponding to any Hopf-Galois object is just H again.

Hopf-Galois objects

- Twisted group algebras, and covariant channels between them, are part of $\mathrm{Chan}(H)$ for a compact quantum group algebra H.
- Hopf-Galois objects for H correspond to 2-cohomology classes $[\psi] \in H^2(G, U(1))$.
- The new compact quantum group algebra corresponding to any Hopf-Galois object is just H again.
- The functor $F_{[\psi]} : \mathrm{Chan}(H) \to \mathrm{Chan}(H)$ maps $A(L, \phi)$ to $A(L, \overline{\psi}\phi)$.

Hopf-Galois objects

- Twisted group algebras, and covariant channels between them, are part of $\mathrm{Chan}(H)$ for a compact quantum group algebra H.
- Hopf-Galois objects for H correspond to 2-cohomology classes $[\psi] \in H^2(G, U(1))$.
- The new compact quantum group algebra corresponding to any Hopf-Galois object is just H again.
- The functor $F_{[\psi]} : \mathrm{Chan}(H) \to \mathrm{Chan}(H)$ maps $A(L, \phi)$ to $A(L, \overline{\psi}\phi)$.
- Covariant channels between twisted group algebras are left unchanged as maps between the underlying vector spaces.

Hopf-Galois objects

- Twisted group algebras, and covariant channels between them, are part of $\mathrm{Chan}(H)$ for a compact quantum group algebra H.
- Hopf-Galois objects for H correspond to 2-cohomology classes $[\psi] \in H^2(G, U(1))$.
- The new compact quantum group algebra corresponding to any Hopf-Galois object is just H again.
- The functor $F_{[\psi]} : \mathrm{Chan}(H) \to \mathrm{Chan}(H)$ maps $A(L, \phi)$ to $A(L, \overline{\psi}\phi)$.
- Covariant channels between twisted group algebras are left unchanged as maps between the underlying vector spaces.
- However, since $F_{[\psi]}$ 'twists' the source and target algebras it will act nontrivially on covariant channels.

Two very concrete examples

- We fix $G := \mathbb{Z}_2 \times \mathbb{Z}_2$.

Two very concrete examples

- We fix $G := \mathbb{Z}_2 \times \mathbb{Z}_2$.
- We consider two twisted group algebras for G:

Two very concrete examples

- We fix $G := \mathbb{Z}_2 \times \mathbb{Z}_2$.
- We consider two twisted group algebras for G:
  - $A(G, 1) \cong \mathbb{C}^{\oplus 4}$.

Two very concrete examples

- We fix $G := \mathbb{Z}_2 \times \mathbb{Z}_2$.
- We consider two twisted group algebras for G:
  - $A(G, 1) \cong \mathbb{C}^{\oplus 4}$.
  - $A(G, \phi_P) \cong B(\mathbb{C}^2)$, where ϕ_P is the 2-cocycle arising from multiplication of Pauli matrices.

Two very concrete examples

- We fix $G := \mathbb{Z}_2 \times \mathbb{Z}_2$.
- We consider two twisted group algebras for G:
  - $A(G, 1) \cong \mathbb{C}^{\oplus 4}$.
  - $A(G, \phi_P) \cong B(\mathbb{C}^2)$, where ϕ_P is the 2-cocycle arising from multiplication of Pauli matrices.
- We will exhibit entanglement-symmetries relating

Two very concrete examples

- We fix $G := \mathbb{Z}_2 \times \mathbb{Z}_2$.
- We consider two twisted group algebras for G:
  - $A(G, 1) \cong \mathbb{C}^{\oplus 4}$.
  - $A(G, \phi_P) \cong B(\mathbb{C}^2)$, where ϕ_P is the 2-cocycle arising from multiplication of Pauli matrices.
- We will exhibit entanglement-symmetries relating
  - Quantum-to-quantum channels $A(G, \phi_P) \to A(G, \phi_P)$ with classical-to-classical channels $A(G, 1) \to A(G, 1)$.

Two very concrete examples

- We fix $G := \mathbb{Z}_2 \times \mathbb{Z}_2$.
- We consider two twisted group algebras for G:
  - $A(G, 1) \cong \mathbb{C}^{\oplus 4}$.
  - $A(G, \phi_P) \cong B(\mathbb{C}^2)$, where ϕ_P is the 2-cocycle arising from multiplication of Pauli matrices.
- We will exhibit entanglement-symmetries relating
  - Quantum-to-quantum channels $A(G, \phi_P) \to A(G, \phi_P)$ with classical-to-classical channels $A(G, 1) \to A(G, 1)$.
  - Quantum-to-classical channels $A(G, \phi_P) \to A(G, 1)$ with classical-to-quantum channels $A(G, 1) \to A(G, \phi_P)$.

Two very concrete examples

- We fix $G := \mathbb{Z}_2 \times \mathbb{Z}_2$.
- We consider two twisted group algebras for G:
    - $A(G, 1) \cong \mathbb{C}^{\oplus 4}$.
    - $A(G, \phi_P) \cong B(\mathbb{C}^2)$, where ϕ_P is the 2-cocycle arising from multiplication of Pauli matrices.
- We will exhibit entanglement-symmetries relating
    - Quantum-to-quantum channels $A(G, \phi_P) \to A(G, \phi_P)$ with classical-to-classical channels $A(G, 1) \to A(G, 1)$.
    - Quantum-to-classical channels $A(G, \phi_P) \to A(G, 1)$ with classical-to-quantum channels $A(G, 1) \to A(G, \phi_P)$.

Covariant channels $A(G,1) \to A(G,1)$

- Covariant channels $A(G,1) \to A(G,1)$ are *weakly symmetric* classical channels with 4 possible inputs and 4 possible outputs.

Covariant channels $A(G,1) \to A(G,1)$

- Covariant channels $A(G,1) \to A(G,1)$ are *weakly symmetric* classical channels with 4 possible inputs and 4 possible outputs.
- They are defined by a stochastic matrix

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{12} & p_{11} & p_{14} & p_{13} \\ p_{13} & p_{14} & p_{11} & p_{12} \\ p_{14} & p_{13} & p_{12} & p_{11} \end{pmatrix}$$

and are therefore determined by a probability distribution $(p_{11}, p_{12}, p_{13}, p_{14})$.

Covariant channels $A(G,1) \to A(G,1)$

- Covariant channels $A(G,1) \to A(G,1)$ are *weakly symmetric* classical channels with 4 possible inputs and 4 possible outputs.
- They are defined by a stochastic matrix

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{12} & p_{11} & p_{14} & p_{13} \\ p_{13} & p_{14} & p_{11} & p_{12} \\ p_{14} & p_{13} & p_{12} & p_{11} \end{pmatrix}$$

  and are therefore determined by a probability distribution $(p_{11}, p_{12}, p_{13}, p_{14})$.
- The classical capacity of one of these channels is

$$C = 2 - H(\{p_{11}, p_{12}, p_{13}, p_{14}\}).$$

Covariant channels $A(G,\phi_P) \to A(G,\phi_P)$

- Recall that $A(G,\phi_P) \cong B(\mathbb{C}^2)$. Recall also the definition of the Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Covariant channels $A(G,\phi_P) \to A(G,\phi_P)$

- Recall that $A(G,\phi_P) \cong B(\mathbb{C}^2)$. Recall also the definition of the Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- A covariant channel $f : A(G,\phi_P) \to A(G,\phi_P)$ is defined by

$$f(I) = I \quad f(X) = \lambda_X X \quad f(Y) = \lambda_Y Y \quad f(Z) = \lambda_Z Z$$

where $\lambda_X, \lambda_Y, \lambda_Z \in [-1,1]$ obey the equations

$$\lambda_X - \lambda_Y + \lambda_Z \leq 1 \qquad \lambda_X + \lambda_Y - \lambda_Z \leq 1$$
$$-\lambda_X + \lambda_Y + \lambda_Z \leq 1 \qquad \lambda_X + \lambda_Y + \lambda_Z \geq -1$$

Covariant channels $A(G, \phi_P) \to A(G, \phi_P)$

- Recall that $A(G, \phi_P) \cong B(\mathbb{C}^2)$. Recall also the definition of the Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- A covariant channel $f : A(G, \phi_P) \to A(G, \phi_P)$ is defined by

$$f(I) = I \quad f(X) = \lambda_X X \quad f(Y) = \lambda_Y Y \quad f(Z) = \lambda_Z Z$$

where $\lambda_X, \lambda_Y, \lambda_Z \in [-1, 1]$ obey the equations

$$\begin{aligned} \lambda_X - \lambda_Y + \lambda_Z &\leq 1 & \lambda_X + \lambda_Y - \lambda_Z &\leq 1 \\ -\lambda_X + \lambda_Y + \lambda_Z &\leq 1 & \lambda_X + \lambda_Y + \lambda_Z &\geq -1 \end{aligned}$$

- These channels scale the Bloch sphere along the X, Y and Z-axes.

The entanglement-symmetry

- Covariant channels $A(G, \phi_P) \to A(G, \phi_P)$ are related to covariant channels $A(G, 1) \to A(G, 1)$ by the Hopf-Galois object $[\phi_P] \in H^2(G, U(1))$.

The entanglement-symmetry

- Covariant channels $A(G, \phi_P) \to A(G, \phi_P)$ are related to covariant channels $A(G, 1) \to A(G, 1)$ by the Hopf-Galois object $[\phi_P] \in H^2(G, U(1))$.
- This entanglement symmetry maps

$$(\lambda_X, \lambda_Y, \lambda_Z) \mapsto \begin{pmatrix} p_{11} = \frac{1}{4}(1 + \lambda_X + \lambda_Y + \lambda_Z) \\ p_{12} = \frac{1}{4}(1 + \lambda_X - \lambda_Y - \lambda_Z) \\ p_{13} = \frac{1}{4}(1 - \lambda_X + \lambda_Y - \lambda_Z) \\ p_{14} = \frac{1}{4}(1 - \lambda_X - \lambda_Y + \lambda_Z) \end{pmatrix}$$

The entanglement-symmetry

- Covariant channels $A(G,\phi_P) \to A(G,\phi_P)$ are related to covariant channels $A(G,1) \to A(G,1)$ by the Hopf-Galois object $[\phi_P] \in H^2(G,U(1))$.
- This entanglement symmetry maps

$$(\lambda_X,\lambda_Y,\lambda_Z) \mapsto \begin{pmatrix} p_{11} = \frac{1}{4}(1+\lambda_X+\lambda_Y+\lambda_Z) \\ p_{12} = \frac{1}{4}(1+\lambda_X-\lambda_Y-\lambda_Z) \\ p_{13} = \frac{1}{4}(1-\lambda_X+\lambda_Y-\lambda_Z) \\ p_{14} = \frac{1}{4}(1-\lambda_X-\lambda_Y+\lambda_Z) \end{pmatrix}$$

- The entanglement-assisted classical capacity of a covariant channel $A(G,\phi_P) \to A(G,\phi_P)$ can therefore be straightforwardly calculated by determining the entropy of the associated probability distribution.

Covariant channels $A(G, \phi_P) \to A(G, 1)$

- Covariant channels $B(\mathbb{C}^2) \cong A(G, \phi_P) \to A(G, 1) \cong \mathbb{C}^{\oplus 4}$ are 4-outcome POVMs on a qubit, defined by positive operators

$$M_I := \frac{1}{4}(I + \lambda_X X + \lambda_Y Y + \lambda_Z Z)$$
$$M_X := \frac{1}{4}(I + \lambda_X X - \lambda_Y Y - \lambda_Z Z)$$
$$M_Y = \frac{1}{4}(I - \lambda_X X + \lambda_Y Y - \lambda_Z Z)$$
$$M_Z := \frac{1}{4}(I - \lambda_X X - \lambda_Y Y + \lambda_Z Z)$$

Covariant channels $A(G, \phi_P) \to A(G, 1)$

- Covariant channels $B(\mathbb{C}^2) \cong A(G, \phi_P) \to A(G, 1) \cong \mathbb{C}^{\oplus 4}$ are 4-outcome POVMs on a qubit, defined by positive operators

$$M_I := \frac{1}{4}(I + \lambda_X X + \lambda_Y Y + \lambda_Z Z)$$
$$M_X := \frac{1}{4}(I + \lambda_X X - \lambda_Y Y - \lambda_Z Z)$$
$$M_Y = \frac{1}{4}(I - \lambda_X X + \lambda_Y Y - \lambda_Z Z)$$
$$M_Z := \frac{1}{4}(I - \lambda_X X - \lambda_Y Y + \lambda_Z Z)$$

- These POVMs are determined by $\vec{\lambda} := (\lambda_X, \lambda_Y, \lambda_Z) \in \mathbb{R}^3$, where $|\vec{\lambda}| \leq 1$.

Covariant channels $A(G,1) \to A(G,\phi_P)$

- Covariant channels $\mathbb{C}^{\oplus 4} \cong A(G,1) \to A(G,\phi_P) \cong B(\mathbb{C}^2)$ are classical-to-quantum channels defined by four density matrices

$$\rho_I := \frac{1}{2}(I + \lambda_X X + \lambda_Y Y + \lambda_Z Z)$$
$$\rho_X := \frac{1}{2}(I + \lambda_X X - \lambda_Y Y - \lambda_Z Z)$$
$$\rho_Y = \frac{1}{2}(I - \lambda_X X + \lambda_Y Y - \lambda_Z Z)$$
$$\rho_Z := \frac{1}{2}(I - \lambda_X X - \lambda_Y Y + \lambda_Z Z)$$

Covariant channels $A(G,1) \to A(G,\phi_P)$

- Covariant channels $\mathbb{C}^{\oplus 4} \cong A(G,1) \to A(G,\phi_P) \cong B(\mathbb{C}^2)$ are classical-to-quantum channels defined by four density matrices

$$\rho_I := \frac{1}{2}(I + \lambda_X X + \lambda_Y Y + \lambda_Z Z)$$
$$\rho_X := \frac{1}{2}(I + \lambda_X X - \lambda_Y Y - \lambda_Z Z)$$
$$\rho_Y = \frac{1}{2}(I - \lambda_X X + \lambda_Y Y - \lambda_Z Z)$$
$$\rho_Z := \frac{1}{2}(I - \lambda_X X - \lambda_Y Y + \lambda_Z Z)$$

- These classical-to-quantum channels are determined by $\vec{\lambda} := (\lambda_X, \lambda_Y, \lambda_Z) \in \mathbb{R}^3$, where $|\vec{\lambda}| \leq 1$.

The entanglement-symmetry

- Covariant channels $A(G, \phi_P) \to A(G, 1)$ are related to covariant channels $A(G, 1) \to A(G, \phi_P)$ by the Hopf-Galois object $[\overline{\phi_P}] \in H^2(G, U(1))$.

The entanglement-symmetry

- Covariant channels $A(G,\phi_P) \to A(G,1)$ are related to covariant channels $A(G,1) \to A(G,\phi_P)$ by the Hopf-Galois object $[\overline{\phi_P}] \in H^2(G,U(1))$.
- This entanglement symmetry maps a point on the Bloch sphere to the opposite point:

$$(\lambda_X, \lambda_Y, \lambda_Z) \mapsto (-\lambda_X, -\lambda_Y, -\lambda_Z)$$

The entanglement-symmetry

- Covariant channels $A(G, \phi_P) \to A(G, 1)$ are related to covariant channels $A(G, 1) \to A(G, \phi_P)$ by the Hopf-Galois object $[\overline{\phi_P}] \in H^2(G, U(1))$.
- This entanglement symmetry maps a point on the Bloch sphere to the opposite point:

$$(\lambda_X, \lambda_Y, \lambda_Z) \mapsto (-\lambda_X, -\lambda_Y, -\lambda_Z)$$

- We see that classical-to-quantum channels and quantum-to-classical channels can be equivalent communication resources in the entanglement-assisted setting.

Future examples

- The construction we have seen so far extends beyond twisted group algebras to any *G-graded* f.d. C^*-algebra.

Future examples

- The construction we have seen so far extends beyond twisted group algebras to any *G-graded* f.d. C^*-algebra.
- These are not the only entanglement-symmetries coming from finite groups — rather than G-gradings we can consider *actions* of G, just as we defined for compact groups earlier.

Future examples

- The construction we have seen so far extends beyond twisted group algebras to any G*-graded* f.d. C^*-algebra.
- These are not the only entanglement-symmetries coming from finite groups — rather than G-gradings we can consider *actions* of G, just as we defined for compact groups earlier.
- In that case Hopf-Galois objects correspond to subgroups $L < G$ which are of *central type*. It is an interesting problem to compute the corresponding functors.

Future examples

- The construction we have seen so far extends beyond twisted group algebras to any G-*graded* f.d. C^*-algebra.
- These are not the only entanglement-symmetries coming from finite groups — rather than G-gradings we can consider *actions* of G, just as we defined for compact groups earlier.
- In that case Hopf-Galois objects correspond to subgroups $L < G$ which are of *central type*. It is an interesting problem to compute the corresponding functors.
- Finite group actions cover natural examples of communication channels, such as uniform noise $f : B(\mathbb{C}^2)^{\otimes n} \to B(\mathbb{C}^2)^{\otimes n}$, i.e. where f is covariant under the permutation action of S_n on the n qubits.

Future examples

- The construction we have seen so far extends beyond twisted group algebras to any *G-graded* f.d. C^*-algebra.
- These are not the only entanglement-symmetries coming from finite groups — rather than G-gradings we can consider *actions* of G, just as we defined for compact groups earlier.
- In that case Hopf-Galois objects correspond to subgroups $L < G$ which are of *central type*. It is an interesting problem to compute the corresponding functors.
- Finite group actions cover natural examples of communication channels, such as uniform noise $f : B(\mathbb{C}^2)^{\otimes n} \to B(\mathbb{C}^2)^{\otimes n}$, i.e. where f is covariant under the permutation action of S_n on the n qubits.

Proof

The tensor category $\mathrm{Corep}(H)$

- For any compact quantum group algebra H we can define its tensor category $\mathrm{Corep}(H)$ of f.d. *unitary corepresentations*.

The tensor category $\mathrm{Corep}(H)$

- For any compact quantum group algebra H we can define its tensor category $\mathrm{Corep}(H)$ of f.d. *unitary corepresentations*.
- This should be thought of as the category of f.d. continuous unitary representations of the associated compact quantum group.

The tensor category $\mathrm{Corep}(H)$

- For any compact quantum group algebra H we can define its tensor category $\mathrm{Corep}(H)$ of f.d. *unitary corepresentations*.
- This should be thought of as the category of f.d. continuous unitary representations of the associated compact quantum group.
- It is a *rigid C^*-tensor category*: in particular, it has

The tensor category $\mathrm{Corep}(H)$

- For any compact quantum group algebra H we can define its tensor category $\mathrm{Corep}(H)$ of f.d. *unitary corepresentations*.
- This should be thought of as the category of f.d. continuous unitary representations of the associated compact quantum group.
- It is a *rigid C^*-tensor category*: in particular, it has
  - A *tensor product*: for any corepresentations V, W we can define the tensor product of corepresentations $V \otimes W$.

The tensor category $\mathrm{Corep}(H)$

- For any compact quantum group algebra H we can define its tensor category $\mathrm{Corep}(H)$ of f.d. *unitary corepresentations*.
- This should be thought of as the category of f.d. continuous unitary representations of the associated compact quantum group.
- It is a *rigid C^*-tensor category*: in particular, it has
  - A *tensor product*: for any corepresentations V, W we can define the tensor product of corepresentations $V \otimes W$.
  - A *dagger* given by the Hermitian adjoint: that is, for any intertwiner of corepresentations $f : V \to W$ there is an adjoint intertwiner $f^\dagger : W \to V$.

From coactions to Frobenius algebras

- An f.d. C^*-algebra A with an H-coaction possesses a canonical H-invariant linear functional $\phi : A \to \mathbb{C}$, which we call the *separable* linear functional.

From coactions to Frobenius algebras

- An f.d. C^*-algebra A with an H-coaction possesses a canonical H-invariant linear functional $\phi : A \to \mathbb{C}$, which we call the *separable* linear functional.
- This linear functional defines an inner product $\langle x|y\rangle := \phi(x^* y)$ on A turning A into a unitary H-corepresentation; that is, an object of $\mathrm{Corep}(H)$.

From coactions to Frobenius algebras

- An f.d. C^*-algebra A with an H-coaction possesses a canonical H-invariant linear functional $\phi : A \to \mathbb{C}$, which we call the *separable* linear functional.
- This linear functional defines an inner product $\langle x|y \rangle := \phi(x^* y)$ on A turning A into a unitary H-corepresentation; that is, an object of $\mathrm{Corep}(H)$.
- This object of $\mathrm{Corep}(H)$ is equipped with multiplication and unit morphisms $m : A \otimes A \to A$ and $u : \mathbb{C} \to A$ satisfying the equations shown on the next slide.

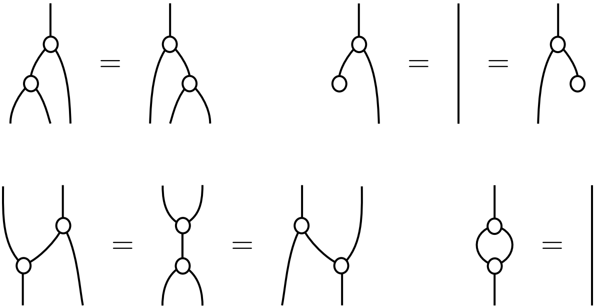
From coactions to Frobenius algebras

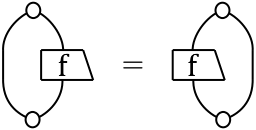
- An f.d. C^*-algebra A with an H-coaction possesses a canonical H-invariant linear functional $\phi : A \to \mathbb{C}$, which we call the *separable* linear functional.
- This linear functional defines an inner product $\langle x|y \rangle := \phi(x^* y)$ on A turning A into a unitary H-corepresentation; that is, an object of $\mathrm{Corep}(H)$.
- This object of $\mathrm{Corep}(H)$ is equipped with multiplication and unit morphisms $m : A \otimes A \to A$ and $u : \mathbb{C} \to A$ satisfying the equations shown on the next slide.
- We can thereby identify f.d. C^*-algebras carrying an H-coaction with *separable standard Frobenius algebras* (SSFAs) in $\mathrm{Corep}(H)$.

From coactions to Frobenius algebras

- An f.d. C^*-algebra A with an H-coaction possesses a canonical H-invariant linear functional $\phi : A \to \mathbb{C}$, which we call the *separable* linear functional.
- This linear functional defines an inner product $\langle x|y\rangle := \phi(x^* y)$ on A turning A into a unitary H-corepresentation; that is, an object of $\mathrm{Corep}(H)$.
- This object of $\mathrm{Corep}(H)$ is equipped with multiplication and unit morphisms $m : A \otimes A \to A$ and $u : \mathbb{C} \to A$ satisfying the equations shown on the next slide.
- We can thereby identify f.d. C^*-algebras carrying an H-coaction with *separable standard Frobenius algebras* (SSFAs) in $\mathrm{Corep}(H)$.
- See S. Neshveyev and M. Yamashita, 'Categorically Morita equivalent compact quantum groups' (2018).

The equations of a SSFA

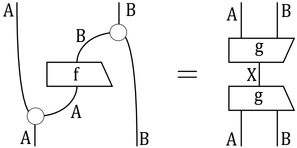




for all $f : A \to A$

From channels to morphisms

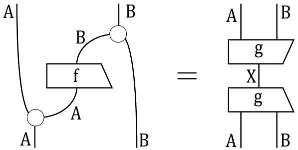
- Covariant completely positive maps $f : A \to B$ correspond to morphisms $f : A \to B$ between the corresponding SSFAs in $\mathrm{Corep}(H)$ satisfying



for some intertwiner $g : A \otimes B \to A \otimes B$. (C. Heunen and J. Vicary, 'Categories for quantum theory' (2019).)

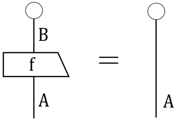
From channels to morphisms

- Covariant completely positive maps $f : A \to B$ correspond to morphisms $f : A \to B$ between the corresponding SSFAs in $\mathrm{Corep}(H)$ satisfying



for some intertwiner $g : A \otimes B \to A \otimes B$. (C. Heunen and J. Vicary, 'Categories for quantum theory' (2019).)

- The CP map $f : A \to B$ preserves the separable functional iff:



A categorical approach to Hopf-Galois theory

- Hopf-Galois objects X for H correspond to functors $F_X : \mathrm{Corep}(H) \to \mathrm{Hilb}$ preserving the tensor product and the dagger. We call these *fibre functors*. (J. Bichon, 'Galois extension for a compact quantum group' (1999).)

A categorical approach to Hopf-Galois theory

- Hopf-Galois objects X for H correspond to functors $F_X : \mathrm{Corep}(H) \to \mathrm{Hilb}$ preserving the tensor product and the dagger. We call these *fibre functors*. (J. Bichon, 'Galois extension for a compact quantum group' (1999).)
- There is a *canonical* fibre functor $F_H : \mathrm{Corep}(H) \to \mathrm{Hilb}$, taking a unitary corepresentation to its underlying Hilbert space and an intertwiner to its underlying linear map.

A categorical approach to Hopf-Galois theory

- Hopf-Galois objects X for H correspond to functors $F_X : \mathrm{Corep}(H) \to \mathrm{Hilb}$ preserving the tensor product and the dagger. We call these *fibre functors*. (J. Bichon, 'Galois extension for a compact quantum group' (1999).)
- There is a *canonical* fibre functor $F_H : \mathrm{Corep}(H) \to \mathrm{Hilb}$, taking a unitary corepresentation to its underlying Hilbert space and an intertwiner to its underlying linear map.
- Finite-dimensional $*$-representations $\pi : X \to B(V)$ of Hopf-Galois objects correspond to *unitary pseudonatural transformations* $u_\pi : F_H \to F_X$. (D. Verdon, 'Unitary transformations of fibre functors' (2022).)

From fibre functors to equivalences

Theorem (Tannaka-Krein-Woronowicz duality)

Let $\mathcal{C}$ be a rigid C^*-tensor category. From any fibre functor $F : \mathcal{C} \to \mathrm{Hilb}$ we obtain:

From fibre functors to equivalences

Theorem (Tannaka-Krein-Woronowicz duality)

Let $\mathcal{C}$ be a rigid C^*-tensor category. From any fibre functor $F : \mathcal{C} \to \mathrm{Hilb}$ we obtain:

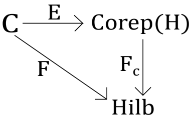
- *A compact quantum group algebra H.*

From fibre functors to equivalences

Theorem (Tannaka-Krein-Woronowicz duality)

Let $\mathcal{C}$ be a rigid C^*-tensor category. From any fibre functor $F : \mathcal{C} \to \mathrm{Hilb}$ we obtain:

- *A compact quantum group algebra H.*
- *An equivalence $E : \mathcal{C} \to \mathrm{Corep}(H)$ preserving the tensor product and the dagger, and making the following diagram of functors commute up to natural isomorphism:*



(S. Neshveyev and L. Tuset, 'Compact quantum groups and their representation categories' (2013), Thm. 2.3.2.)

Obtaining the theorem

- We just saw that $\mathrm{Chan}(H)$ can be expressed as a category of SSFAs in $\mathrm{Corep}(H)$.

Obtaining the theorem

- We just saw that $\mathrm{Chan}(H)$ can be expressed as a category of SSFAs in $\mathrm{Corep}(H)$.
- We also saw that Hopf-Galois objects X for H yield new compact quantum group algebras H^X and tensor-preserving equivalences $F_X : \mathrm{Corep}(H) \to \mathrm{Corep}(H^X)$.

Obtaining the theorem

- We just saw that $\mathrm{Chan}(H)$ can be expressed as a category of SSFAs in $\mathrm{Corep}(H)$.
- We also saw that Hopf-Galois objects X for H yield new compact quantum group algebras H^X and tensor-preserving equivalences $F_X : \mathrm{Corep}(H) \to \mathrm{Corep}(H^X)$.
- These tensor-preserving equivalences map SSFAs to SSFAs and channels to channels, yielding the induced equivalence $F_X : \mathrm{Chan}(H) \to \mathrm{Chan}(H^X)$.

Obtaining the theorem

- We just saw that $\mathrm{Chan}(H)$ can be expressed as a category of SSFAs in $\mathrm{Corep}(H)$.
- We also saw that Hopf-Galois objects X for H yield new compact quantum group algebras H^X and tensor-preserving equivalences $F_X : \mathrm{Corep}(H) \to \mathrm{Corep}(H^X)$.
- These tensor-preserving equivalences map SSFAs to SSFAs and channels to channels, yielding the induced equivalence $F_X : \mathrm{Chan}(H) \to \mathrm{Chan}(H^X)$.
- The rest follows from the correspondence between f.d. $*$-representations of Hopf-Galois objects and unitary pseudonatural transformations of fibre functors.

Conclusion

Review

- We raised the problem of classifying channels which are *entanglement-equivalent*. Such channels are equally powerful communication resources when the transmitter and receiver share quantum entanglement.

Review

- We raised the problem of classifying channels which are *entanglement-equivalent*. Such channels are equally powerful communication resources when the transmitter and receiver share quantum entanglement.
- We showed that if a channel is covariant with respect to actions of some compact (quantum) group G, then Hopf-Galois objects for G can be used to construct entanglement-equivalent channels.

Review

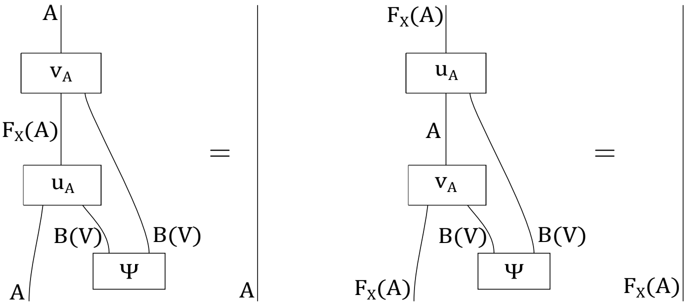
- We raised the problem of classifying channels which are *entanglement-equivalent*. Such channels are equally powerful communication resources when the transmitter and receiver share quantum entanglement.
- We showed that if a channel is covariant with respect to actions of some compact (quantum) group G, then Hopf-Galois objects for G can be used to construct entanglement-equivalent channels.
- How does this relate to prior constructions?

An observation about our construction

- Note that the encoding and decoding channels

$$u_A : A \otimes B(V) \to F_X(A) \qquad v_A : F_X(A) \otimes B(V) \to A$$

arising from a Hopf-Galois object X obey the following equations (because functors preserve identity morphisms):

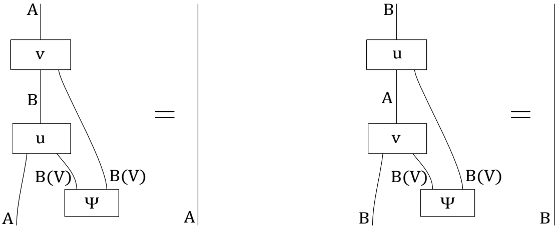


Entanglement-invertible channels: I

- Let's forget about the group theory and imagine we just have a pair of channels

$$u : A \otimes B(V) \to B \qquad v : B \otimes B(V) \to A$$

satisfying these equations:

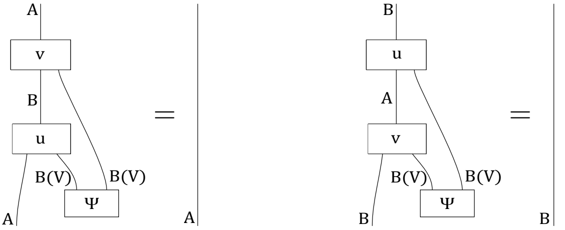


Entanglement-invertible channels: I

- Let's forget about the group theory and imagine we just have a pair of channels

$$u : A \otimes B(V) \to B \qquad v : B \otimes B(V) \to A$$

satisfying these equations:



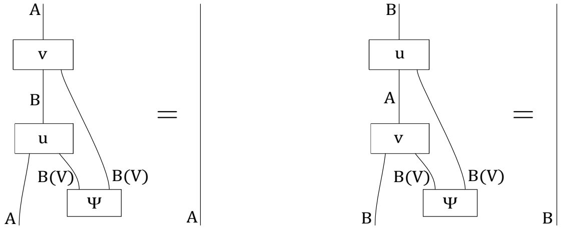
- Werner 'All teleportation and dense coding schemes' (2001).

Entanglement-invertible channels: I

- Let's forget about the group theory and imagine we just have a pair of channels

$$u : A \otimes B(V) \to B \qquad v : B \otimes B(V) \to A$$

satisfying these equations:



- Werner 'All teleportation and dense coding schemes' (2001).
- D. Verdon, 'Entanglement-invertible channels' (2022).

A simple construction

- Now for any channel $f : B \to B$, if

$$u \circ v \circ f \circ u \circ v = f$$

then the channels f and $v \circ f \circ u$ are entanglement-equivalent.

A simple construction

- Now for any channel $f : B \to B$, if

$$u \circ v \circ f \circ u \circ v = f$$

  then the channels f and $v \circ f \circ u$ are entanglement-equivalent.
- This construction was used in C.H. Bennett et al., 'Entanglement-assisted capacity of noisy quantum channels' (1999).

A simple construction

- Now for any channel $f : B \to B$, if

$$u \circ v \circ f \circ u \circ v = f$$

  then the channels f and $v \circ f \circ u$ are entanglement-equivalent.
- This construction was used in C.H. Bennett et al., 'Entanglement-assisted capacity of noisy quantum channels' (1999).
- Our construction:
  - makes explicit the group theory behind this simple construction.

A simple construction

- Now for any channel $f : B \to B$, if

$$u \circ v \circ f \circ u \circ v = f$$

  then the channels f and $v \circ f \circ u$ are entanglement-equivalent.
- This construction was used in C.H. Bennett et al., 'Entanglement-assisted capacity of noisy quantum channels' (1999).
- Our construction:
  - makes explicit the group theory behind this simple construction.
  - generalises it to channels with different source and target.

A simple construction

- Now for any channel $f : B \to B$, if

$$u \circ v \circ f \circ u \circ v = f$$

  then the channels f and $v \circ f \circ u$ are entanglement-equivalent.
- This construction was used in C.H. Bennett et al., 'Entanglement-assisted capacity of noisy quantum channels' (1999).
- Our construction:
  - makes explicit the group theory behind this simple construction.
  - generalises it to channels with different source and target.
  - shows that these are coherent transformations of whole categories of covariant channels, not just of a single channel.

Two questions for the future

- Is it possible to find new constructions of entanglement-equivalent channels that do not arise from covariance in the way we have outlined here?

Two questions for the future

- Is it possible to find new constructions of entanglement-equivalent channels that do not arise from covariance in the way we have outlined here?
- Is it possible to classify entanglement-equivalence classes of channels in general?

Thanks for listening!